Nonnegative Solutions for Dominant Diagonal Matrices with Both Signs in the Off-Diagonals

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This paper considers a well-known linear system that is in widespread use. Any comparative statics exercise that is often employed in economics can be represented by a linear system. In order to guarantee that the solution to this linear system is nonnegative, previous research assumed that the coefficient matrix in this linear system was a Metzler matrix, i.e. each off-diagonal entry of this matrix was nonnegative. In this paper we relax this assumption, and apply this mathematical technique to study the profit-maximization pricing of a multi-product monopolist.

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JEL classification: C62, L12, L32

1 Introduction

This paper studies the following linear system:

$$Ax = b, \tag{1.1}$$

where $A = [a_{ij}]$ is an $n \times n$ matrix, and both x and b are $n \times 1$ vectors. In the

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literature, A is usually assumed to have a negative dominant diagonal.[1] Under this assumption, it is known that each element of x is nonnegative if each off-diagonal entry of A is nonnegative (i.e. $a_{ij} \geq 0$ for all i and j if $i \neq j$), and if each element of b is nonpositive (see Proposition 2.1 below). Simon (1989) extended this result by allowing b to have both signs in its entries (see Proposition 2.2 below). This paper aims at further extending Simon's result by allowing the off-diagonal entries of A to have both signs. In other words, this paper aims at developing a mathematical method which is capable of studying cases in which both the entries of b and the off-diagonal entries of A are allowed to have both signs.

The above linear system is actually in widespread use. For example, Simon (1989, p. 217) has already noted that both Leontief's input-output analysis of economic systems and the Stolper-Samuelson study of world goods' prices can be represented by the above linear system.

More importantly, we argue that each comparative statics problem can also be represented by the linear system. For example, consider the following non-linear dynamic system:

$$\frac{dy}{dt} = f(y(t), \theta), \tag{1.2}$$

where y is an $n \times 1$ vector, t stands for time, θ is a parameter, and f is an $n \times 1$ vector-valued function. The equilibrium condition for this system is:

$$f(y, \theta) = 0. \tag{1.3}$$

The comparative statics problem which studies the response of the equilibrium value of y with respect to θ is represented by:

$$\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial \theta}, \tag{1.4}$$

where both $\partial f/\partial y$, an $n \times n$ matrix, and $\partial f/\partial \theta$, an $n \times 1$ vector, are evaluated at the equilibrium value of y, and $\partial y/\partial \theta$ is an $n \times 1$ vector to be solved. Equation (1.4) can be interpreted as the linear system (1.1): $\partial f/\partial y$ corresponds to the square matrix A, $\partial y/\partial \theta$ to x, and $-\partial f/\partial \theta$ to b.

It remains to be shown that, in studying the linear system (1.4), it is not unusual to assume that the matrix $\partial f/\partial y$ has a negative dominant diagonal. Using some regularity conditions specified in Beavis and Dobbs (1990, Theorem 5.19),

[1] A square matrix A is said to have a dominant diagonal if $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$ for every j, where $|a_{ij}|$ stands for the modulus of a_{ij}. Furthermore, A is said to have a negative dominant diagonal if A has a dominat diagonal and each diagonal entry of A is strictly negative.

the behavior of solutions to the nonlinear dynamic system (1.2) is similar to that of the following linear dynamic system:

$$\frac{dy}{dt} = \frac{\partial f}{\partial y}\left(y - \hat{y}\right),$$

in a sufficiently small neighborhood of $\hat{y}$, where $\hat{y}$ is an equilibrium point of y, $\hat{y}$ is unique in a certain open ball about $\hat{y}$, and $\partial f/\partial y$ is evaluated at $\hat{y}$. Therefore, the sufficient and necessary condition for the non-linear dynamic system to be asymptotically locally stable is that $\partial f/\partial y$ should be a stable matrix (Beavis and Dobbs, 1990, p. 151). Furthermore, for $\partial f/\partial y$ to have a negative dominant diagonal is a sufficient condition for ensuring that $\partial f/\partial y$ is a stable matrix (Beavis and Dobbs 1990, Theorem 5.30). Therefore, it is not unusually assumed that $\partial f/\partial y$ has a negative dominant diagonal in order to conduct the comparative statics analysis since it is meaningless to conduct the comparative statics analysis if the system is not asymptotically locally stable.

If the linear system (1.1) is regarded as a comparative statics system, then it is not hard to imagine a case where the off-diagonal entries of A actually have both signs. Therefore, it is important to provide a mathematical technique to study this case. This is the main purpose of this paper.

Let us consider the following specific dynamic system to strengthen our arguments:

$$\frac{dp(t)}{dt} = f(p(t), \theta),$$

where $p(t)$ is an $n \times 1$ vector in which the ith element is the time path of the price of the ith commodity, and $f(p(t), \theta)$ is an $n \times 1$ vector in which the ith element is the excess demand function for the ith commodity. Let $\hat{p}$ be an equilibrium price vector so that $f(\hat{p}, \theta) = 0$ and $\hat{p}$ is unique in a certain open ball about $\hat{p}$. In order to utilize (1.1) to study the response of the equilibrium value of p with respect to θ, we define $a_{ij} \equiv \partial f_i/\partial p_j$ which is evaluated at $p = \hat{p}$.

In this case, the "negative dominant diagonal" has a natural economic interpretation: each own-price effect is negative and its modulus dominates the sum of the moduli of its corresponding cross-price effects.

The positivity of a_{ij} $(i \neq j)$ means that the excess demand for the ith commodity rises when the price of the jth commodity rises. Therefore, the assumption of positive off-diagonal entries means that all commodities are gross substitutes for each other. However, in reality, it is not uncommon for there to be complementarities between some commodities. Therefore, in this context, it is important to allow off-diagonal entries of A to have both signs.

Equation (1.3) can be reinterpreted as the first-order necessary condition of a maximization problem involving an economic unit. In this case, Equation (1.4) represents the comparative statics problem that characterizes how the economic unit responds to a change in its economic environment. In this context, $\partial f/\partial y$ is required to be a negative semi-definite matrix so that the second-order necessary condition of the maximization problem will hold. Therefore, in general, $\partial f/\partial y$ automatically has nonnegative diagonals. Furthermore, in Section 3 we will study a maximization problem where it is not unusual to assume that $\partial f/\partial y$ has a dominant diagonal.

This paper is organized as follows. Section 2 presents the mathematical analysis of the nonnegativity of x. Section 3 applies mathematical results to study the profit-maximization pricing of a multi-product monopolist. In this example, it is assumed that there are complementarities between some of the goods produced by the monopolist. This feature will make matrix A have both signs in its off-diagonal entries. Moreover, it is known in the literature that this feature may give rise to some perverse results (see Section 3), and hence it is interesting to study this case.

2 Sufficient Conditions for Nonnegative Solutions

In this section, we begin by summarizing the results already established in the literature (Propositions 2.1 and 2.2) before proceeding to present our new propositions (Proposition 2.3, Corollary 2.1 and Proposition 2.4).

If A is a square matrix with a negative dominant diagonal and if each off-diagonal entry of A is nonnegative, then from Theorem 4.D.3 of Takayama (1985), it follows that A is nonsingular and its inverse is a nonpositive matrix, that is, each entry of A^{-1} is nonpositive. Therefore we have the following proposition:

Proposition 2.1. *Assume that A has a negative dominant diagonal. The solution x of the linear system (1.1) is nonnegative if the following conditions hold:*

- *(i) each off-diagonal entry of A is nonnegative, i.e. $a_{ij} \geq 0$ for all i and j $(i \neq j)$,*
- *(ii) $b \leq 0$, i.e. $b_i \leq 0$ for all i.*

Condition *(ii)* of the above proposition requires that each entry of b should be nonpositive. Simon (1989) relaxed this condition:

Proposition 2.2. *(Simon 1989, Theorem 1) Assume that A has a negative dominant diagonal. The solution x_i of the linear system (1.1) is nonnegative if the following conditions hold:*

(i) *each off-diagonal entry of A is nonnegative,*

(ii) $b_i \leq 0$, *and the modulus of b_i is not smaller than the sum of positive elements in the vector b, i.e.* $|b_i| \geq \sum_{h \in B} |b_h|$.

Here $B = \{h; b_h > 0\}$ is an index set for positive components in b. Based on this definition, $\sum_{h \in B} |b_h|$ is equal to $\sum_{h \in B} b_h$.

Note that this proposition allows some elements of b to be positive. However, in order to ensure that x_i is nonnegative, Condition *(ii)* requires that these positive elements should be "dominated" by b_i.

After summarizing the results established in the literature, let us proceed to relax Condition (i) of Proposition 2.2. However, let us begin by defining two sets of indices $R_i = \{k : a_{ik} < 0, k \neq i\}$ and $C^j = \{h : a_{hj} < 0, h \neq j\}$ for the negative off-diagonal entries in the ith row vector a_i and the jth column vector a^j, respectively. The following proposition deals with a case where some off-diagonal entries in A are negative, but the off-diagonal entries in the ith row vector a_i are still all nonnegative:

Proposition 2.3. *Consider the case where all off-diagonal entries in a_i are nonnegative, i.e. $R_i = \emptyset$. Assume that $a_{jj} < 0$ and $|a_{jj}| > \sum_{i \notin C^j} |a_{ij}|$ for each j. The solution x_i of the linear system (1.1) is nonnegative if the following conditions hold:*

(i) $a_{ij} \geq \sum_{h \in C^j} |a_{hj}|$ *for every* $j \neq i$.

(ii) $b_i \leq 0$, *and the modulus of b_i is not smaller than the sum of positive elements in the vector b.*

Proof. See the Appendix. □

Condition *(ii)* of this proposition is the same as that of Proposition 2.2. Furthermore, Condition (i), like Condition (ii), also contains an idea of "dominance": Condition (i) requires that the negative off-diagonal entries in the jth column vector should be "dominated", in modulus, by a_{ij}. In other words, Condition (i) means that "perverse" (i.e. negative) off-diagonal entries should be dominated by "normal" (i.e. nonnegative) off-diagonal entries.

Note that the assumption "$|a_{jj}| > \sum_{i \notin C^j} |a_{ij}|$ for each j" is less restrictive than the assumption "$|a_{jj}| > \sum_{i \neq j} |a_{ij}|$ for each j". The next corollary is therefore an immediate consequence of Proposition 2.3.

Corollary 2.1. *Consider the case where all off-diagonal entries in a_i are nonnegative. Assume that A has a negative dominant diagonal. The solution x_i of the linear system (1.1) is nonnegative if the following conditions hold:*

(i) $a_{ij} \geq \sum_{h \in C^j} |a_{hj}|$ *for every* $j \neq i$.

(ii) $b_i \leq 0$, *and the modulus of* b_i *is not smaller than the sum of positive elements in the vector* b.

In the above corollary, all off-diagonal entries in a_i are assumed to be nonnegative. In the next proposition, off-diagonal entries in a_i are allowed to have both signs:

Proposition 2.4. *Consider the case where some of the off-diagonals in* a_i *are negative, i.e.* $R_i \neq \emptyset$. *Assume that A has a negative dominant diagonal. The solution* x_i *of the linear system (1.1) is nonnegative if the following conditions hold:*

(i) $a_{ij} \geq \sum_{h \in C^j \cup R_i} |a_{hj}|$ *for every* $j \notin R_i$ *and* $j \neq i$. *However, the inequality should be strict if* $\sum_{h \in C^j \cup R_i} |a_{hj}| > 0$; *and*

(ii) $b_i \leq 0$, *and* $|b_i| \geq \sum_{h \in B \cup R_i} |b_h|$. *However, the inequality should be strict if* $\sum_{h \in B \cup R_i} |b_h| > 0$.

Proof. See the Appendix. □

Roughly speaking, Condition (i) means that each "normal" (i.e., nonnegative) off-diagonal entry of a_i should be large enough, in modulus. Condition (ii) has a similar meaning.

It is important to emphasize a difference between Condition (i) of this proposition and Condition (i) of Proposition 2.3. Note that a_{hj} is nonnegative if $h \notin C^j$ but $h \in R_i$. Therefore, the set $\{a_{hj} | h \in C^j \cup R_i\}$ may contain some "normal" off-diagonal entries. Condition (i) of Proposition 2.4 requires that each such normal off-diagonal entry cannot be too large, i.e. a_{hj} cannot be too large if $h \notin C^j$ but $h \in R_i$. However, in Condition (i) of Proposition 2.3, an off-diagonal entry of A should be required to be small enough in modulus only when it is abnormal (i.e. negative).

We may wonder why the above normal off-diagonal entries of A should be regulated. Section 3 will provide economic intuition for this requirement in a specific economic context.

Similarly, the set $\{b_h | h \in B \cup R_i\}$ may contain "normal" (i.e. nonpositive) entries of b since b_h is nonpositive if $h \notin B$ but $h \in R_i$. Condition (ii) requires that any such normal entry of b cannot be too large in modulus.

3 Profit-maximization Pricing of a Multi-product Monopolist

In this section, we apply the mathematical results established in Section 2 to study the profit-maximization pricing of a multi-product monopolist. In this example, it is assumed that there are complementarities between some of the goods produced by the monopolist. This feature will cause matrix A to have both signs in its off-diagonal entries. In studying this example, we will provide economic intuition in relation to the technical conditions stated in Section 2.

In theory, a multi-product monopolist has two alternative pricing rules. One is the naïve profit-maximization pricing rule, while the other is Ramsey pricing which aims at maximizing consumers' welfare subject to a break-even constraint (e.g., Bös 1989, Ch.8). Naïve profit-maximization pricing itself is not interesting since, in reality, every monopolist is regulated by the government in some way. However, it is known that both pricing rules share the same price structure, even though price levels differ (Bös 1989, p. 189; Chang 1996, p. 285; Chang and Chang 2001, p. 367). Accordingly, in order to avoid unnecessary complexities, it is better to analyze profit-maximization pricing than to directly study Ramsey pricing, since what we care about is the price structure rather than the price level when the pricing rule of a multi-product monopolist is examined. This is exactly the approach utilized by Chang (1996) and Chang and Chang (2001) to simplify the analysis of Ramsey pricing. Furthermore, it should be noted that Ramsey pricing is actually closely related to network-access pricing. For example, Section 6 of Chang (1996) demonstrated that a typical network-access pricing problem was in essence a Ramsey pricing problem.

Imagine that a multi-product monopolist produces goods $q = (q_1, \cdots, q_n)$, charges prices $p = (p_1, \cdots, p_n)$, and that the cost of producing output vector q is $C(q)$. Therefore, the profit function is

$$\sum_{i=1}^{n} p_i q_i(p) - C(q(p)), \tag{3.1}$$

where $q_i(p)$ is the demand function for good i.

The first-order necessary condition for the profit-maximization problem can be written as follows:

$$q_i(p) + \sum_{j=1}^{n} \left(p_j - mc_j\right) \frac{\partial q_j}{\partial p_i}(p) = 0, \quad \text{for all } i, \tag{3.2}$$

where $mc_j \equiv \partial C(q)/\partial q_j$ is the marginal cost of producing q_j. The first-order condition (3.2) can be rewritten as the linear system $Ax = b$ if we state that $a_{ij} = \partial q_j/\partial p_i, x_i = p_i - mc_i$, and $b_i = -q_i$.

In this context, some of the off-diagonal entries of A are automatically non-positive if there are gross complementarities between some goods. Furthermore, throughout this section, it is assumed that $\partial q_j/\partial p_j < 0$ for all j, and hence each diagonal entry of A is negative. Therefore, A has a negative dominant diagonal if each own price effect dominates its corresponding cross price effects in the sense that $|\partial q_j/\partial p_j| > \sum_{k\neq j} |\partial q_k/\partial p_j|$ for all j.

3.1 Nonnegative Profit Margins

The term $p_i - mc_i$ is referred to as the profit margin of good i. It should be noted that in Ramsey pricing literature it is important to determine the signs of the profit margins (Chang 1996, footnote 2; Chang and Chang 2001, p. 367). Chang and Chang (2001) studied a case where the goods are gross substitutes for each other, and established sets of sufficient conditions that guarantee that Ramsey prices exceed their corresponding marginal costs. However, it is known that if a good is a complement to some of the other goods, then its profit margin tends to be lower (Tirole 1988, p. 70), and hence its profit margin is more likely to be negative (Bös 1989, p. 200; Chang and Chang 2001, p. 366). Therefore, it is important to find sufficient conditions that can guarantee that such a profit margin will be nonnegative in a case where gross complementarities in fact exist.

Note that, in the context of the profit-maximization problem, R_i is an index set for the complements of good i, and C^j is an index set for the complements of good j.[2] Because every element of b is non-positive, $B = \emptyset$. Therefore, applying Proposition 2.3 to the profit-maximization problem straightforwardly yields the following corollary:

Corollary 3.1. *Consider the case where good i is a substitute for each of the other goods in the sense that $\partial q_j/\partial p_i \geq 0$ for every $j \neq i$. Assume that $\partial q_j/\partial p_j < 0$ and that $|\partial q_j/\partial p_j| > \sum_{k\notin C^j} |\partial q_k/\partial p_j|$ for each j. The profit margin of good i is nonnegative if the following condition holds:*

(i) $\partial q_j/\partial p_i \geq \sum_{h\in C^j} |\partial q_j/\partial p_h|$ *for every* $j \neq i$.

In a nutshell, Condition *(i)* means that the effect caused by the complementarities between the goods is relatively small. Because this complementarity effect is the only effect noted in the literature which may lead to a perverse result, it is

[2]Due to income effects, $\partial q_j/\partial p_i$ and $\partial q_i/\partial p_j$ may have different signs ($i \neq j$). If $\partial q_j/\partial p_i$ and $\partial q_i/\partial p_j$ share the same sign for every $i \neq j$, then $C^k = R_k$ for every k.

intuitively plausible to utilize this condition to ensure that the profit margin of good i is positive.

Applying Proposition 2.4 to the profit-maximization problem yields the following corollary:

Corollary 3.2. *Consider the case where good i is a complement to some of the goods produced, i.e. $R_i \neq \emptyset$. Assume that $|\partial q_j/\partial p_j| > \sum_{k\neq j} |\partial q_k/\partial p_j|$ for all j. The profit margin of good i is nonnegative if the following two conditions hold:*

(i) $\partial q_j/\partial p_i > \sum_{h\in C^j \cup R_i} |\partial q_j/\partial p_h|$ *for every $j \notin R_i$ and $j \neq i$; and*

(ii) $q_i > \sum_{h\in R_i} q_h$.

Condition *(i)* of this corollary is similar to that of Corollary 3.1. However, there is an important difference. Note that $\partial q_j/\partial p_h$ is nonnegative if $h \notin C^j$ but $h \in R_i$, i.e. good h is a substitute for good j, but a complement to good i. Condition *(i)* of this corollary requires that each such nonnegative cross-price effect cannot be too large. In other words, Condition *(i)* of this corrollary requires that $\partial q_j/\partial p_h$ cannot be too large in modulus if good h is a complement to either good j or good i. By contrast, the modulus of $\partial q_j/\partial p_h$ should be regulated only when good h is a complement to good j.

Before proceeding to interpret Condition *(ii)*, let us take a closer look at what may motivate the monopolist to price a good below its marginal cost. The main motivation to price a good below its marginal cost is that this low price raises the demand for those goods which are complements for the good, and hence the profit sacrificed by lowering the price of the good can be compensated for by the profit gained from increasing the demand for its complements.

Condition *(ii)* means that it is less worthwhile pricing a good below its marginal cost if the demand for this good itself is high enough, relatively speaking. This is intuitively plausible since it is not easy to compensate for the profit sacrificed by a good for which demand is relatively high. Condition *(ii)* therefore implies that it is less likely that a good for which demand is high will be cross-subsidized.[3]

Compared with the literature, Corollaries 3.1 and 3.2 are new results.[4]

[3] A good is said to be cross-subsidized by the other goods if its profit margin is negative while the profit margins of the other goods are positive.

[4] According to Proposition 2.1 of Chang and Chang (2001), the profit margins are nonnegative if either one of the following conditions holds: (i) all products produced by the monopolist are compensated substitutes for each other; (ii) the products produced by the monopolist are quasi-separable from the other goods; (iii) the products produced by the monopolist are weakly separable from the other goods and the sub-utility function of the products produced by the monopolist is homothetic. Our Corollaries 3.1 and 3.2 cannot be derived from this proposition.

In what follows, we utilize a more specific example to further illustrate the above two corollaries. In particular, this specific example aims at demonstrating the economic intuition behind Condition (i) of Corollary 3.2 since this condition seems quite complex. Assume that the goods produced by the monopolist can be divided into two groups — Group 1 contains goods 1 and 2, and Group 2 contains goods 3, 4, $\cdots$, n. Goods 1 and 2 are gross complements for each other. Any two goods in Group 2 are gross substitutes for each other. Moreover, good i and good j are gross substitutes for each other if they do not belong to the same group. In short, only $\partial q_2/\partial p_1$ and $\partial q_1/\partial p_2$ are negative in all of the off-diagonal entries. We also suppose that matrix A has a negative dominant diagonal.

Note that $C^j = \emptyset$ if $j \geq 3$. Corollary 3.1 implies that, for any good i belonging to Group 2, its profit margin is nonnegative if $\partial q_i/\partial p_1 \geq |\partial q_2/\partial p_1|$ and $\partial q_i/\partial p_2 \geq |\partial q_1/\partial p_2|$. It is easy to interpret these two conditions.

Corollary 3.2 is next applied to derive sufficient conditions of nonnegative profit margins for goods in Group 1. Condition (ii) implies that only one good in Group 1 can be guaranteed to have a nonnegative profit margin. For example, if $q_1 > q_2$, then it is clear that only good 1 satisfies Condition (ii).

Let us assume that $q_1 > q_2$. Condition (i) implies that the profit margin of good 1 is nonnegative if $\partial q_j/\partial p_1 \geq \partial q_j/\partial p_2$ for every $j \geq 3$. The positive externality noted by Tirole (1988, p. 70) can be applied to explain the above condition. Raising the price of any good in Group 1 generates, based on the definition of gross substitutability, a positive externality to each good in Group 2 in the sense that it raises the demand for each good in Group 2. Condition (ii) implies that raising the price of good 1 generates a greater positive externality than raising the price of good 2, and hence it is more desirable to raise the price of good 1.

3.2 Comparative Statics

This subsection focuses on how an exogenous change in demand affects the optimal profit-maximization prices. For example, this subsection aims to address the issue of whether the price of good i should be increased if the demand for good i is exogenously raised.

Each entry of b is nonpositive in each case studied in Subsection 3.1. By contrast, in this subsection we will study a case in which b has both signs in its entries. Let $q(p, \theta)$ be the Marshallian demand for q where, in the notation, θ is emphasized to be an exogenous parameter affecting q. Therefore, the profit function is:

$$\Pi(p,\theta) = \sum_{j=1}^{n} p_j q_j(p,\theta) - C(q(p,\theta)).$$

The first-order necessary condition for profit-maximization is $\partial\Pi(p,\theta)/\partial p = 0$. The comparative statics system which characterizes how the optimal price vector responds to a change in θ is as follows:

$$\frac{\partial^2\Pi}{\partial p^2}\frac{dp}{d\theta} = -\frac{\partial^2\Pi}{\partial\theta\partial p}. \tag{3.3}$$

Note that the second-order necessary condition of the profit-maximization problem requires $\partial^2\Pi/\partial p^2$ to be negative semi-definite.

The sign of $dp/d\theta$ can be unambiguously determined if only a single good is produced by the monopolist. In this single-product case, $\partial^2\Pi/\partial p^2$ is a non-positive scalar. Therefore, from (3.3) it follows that $dp/d\theta$ shares the same sign as $\partial^2\Pi/\partial\theta\partial p$ (except when $\partial^2\Pi/\partial p^2$ happens to be zero, an extreme case which we do not want to consider). Accordingly, the optimal price should be raised (i.e. $dp/d\theta \geq 0$) if $\partial^2\Pi/\partial\theta\partial p \geq 0$.

Similarly, it is easy to show that the sign of $dp_i/d\theta$ can be unambiguously determined (namely, $dp_i/d\theta$ shares the same sign as $\partial^2\Pi/\partial\theta\partial p_i$) if all of the demand functions are independent of each other (i.e. $\partial q_j/\partial p_i = 0$ for every $j \neq i$).

We next consider a case where three products are produced by the monopolist. It might be convenient for us to work with a cost function $C(q)$ that takes a linear form as defined by

$$C(q) = c_1q_1 + c_2q_2 + c_3q_3,$$

where c_i is the constant marginal cost of producing q_i. The demand functions are assumed to be as follows:

$$\begin{aligned} q_1(p,\theta) &= g_1(\theta) + \delta_{11}p_1 + \delta_{12}p_2 + \delta_{13}p_3, \\ q_2(p,\theta) &= g_2(\theta) + \delta_{21}p_1 + \delta_{22}p_2 + \delta_{23}p_3, \\ q_3(p,\theta) &= g_3(\theta) + \delta_{31}p_1 + \delta_{32}p_2 + \delta_{33}p_3, \end{aligned}$$

where intercept $g_i(\theta)$, in the notation, is emphasized to depend upon θ, and the coefficient δ_{ij} is a constant. Under these assumptions, Equation (3.3) can be expressed as a linear system $Ax = b$:

$$A \equiv \frac{\partial^2\Pi}{\partial p^2} = \begin{pmatrix} 2\delta_{11} & \delta_{12}+\delta_{21} & \delta_{13}+\delta_{31} \\ \delta_{21}+\delta_{12} & 2\delta_{22} & \delta_{23}+\delta_{32} \\ \delta_{31}+\delta_{13} & \delta_{32}+\delta_{23} & 2\delta_{33} \end{pmatrix},$$

$$x \equiv \frac{dp}{d\theta} = \begin{pmatrix} dp_1/d\theta \\ dp_2/d\theta \\ dp_3/d\theta \end{pmatrix},$$

$$b \equiv -\frac{\partial^2 \Pi}{\partial\theta\partial p} = \begin{pmatrix} -\partial g_1(\theta)/\partial\theta \\ -\partial g_2(\theta)/\partial\theta \\ -\partial g_3(\theta)/\partial\theta \end{pmatrix}.$$

Note that, in this case, A is a symmetric matrix.

Suppose that, for each commodity, the own-price effect exceeds, in modulus, the sum of its cross-price effects, i.e. $|\delta_{jj}| > \sum_{i \neq j} |\delta_{ij}|$ for every $j = 1, 2, 3$, and $|\delta_{ii}| > \sum_{j \neq i} |\delta_{ij}|$ for every $i = 1, 2, 3$. It follows that A has a negative dominant diagonal.

We next study whether the prices should be increased if the demand for each good is increased (i.e. $\partial g_i(\theta)/\partial\theta \geq 0$ for all i). If all of the goods are gross substitutes for each other, then each off-diagonal entry of $A$is nonnegative. Proposition 2.1 thus implies that, in this pure substitution case, each price should be raised (i.e. $dp_i/d\theta \geq 0$ for all i) if the demand for each good is increased. The intuition runs as follows. Recall that each price should be raised when all of the demand functions are independent of each other. Furthermore, gross substitutabilities between the goods will reinforce the incentive to raise prices since an increase in the price of one good will further increase the demand for each of the other goods.

Hereafter, we consider a case where there exist complementarities between the goods. It is assumed that all of the goods are gross substitutes for each other except that good 2 and good 3 are gross complements for each other; i.e. $\delta_{23} < 0$, $\delta_{32} < 0$, $\delta_{1j} \geq 0$ $(j = 2, 3)$, and $\delta_{i1} \geq 0$ $(i = 2, 3)$. Note that, in this case, an increase in demand for one commodity may decrease the demand for another good. For example, if an exogenous increase in demand for good 2 causes its price to rise, then, based on the definition of gross complementarity, the demand for good 3 will fall.

In short, if there exist gross complementarities, then some optimal prices might fall even when the demand for each good is raised. Therefore, it is important to determine whether prices should be raised in such a case. We now have the following corollary:

Corollary 3.3. *Consider the case where there exist three goods, and the demand for each good is exogenously raised, i.e.* $\partial g_i(\theta)/\partial\theta \geq 0$ *for every* $i = 1, 2, 3$. *Assume that* $|\delta_{jj}| > \sum_{i \neq j} |\delta_{ij}|$ *for every* $j = 1, 2, 3$, *and* $|\delta_{ii}| > \sum_{j \neq i} |\delta_{ij}|$ *for every* $i = 1, 2, 3$. *We have*

(i) $dp_1/d\theta \geq 0$ *if the following condition holds:*

(a) $a_{12} \equiv \delta_{12} + \delta_{21} > |\delta_{23} + \delta_{32}| \equiv |a_{32}|$ *and* $a_{13} \equiv \delta_{13} + \delta_{31} > |\delta_{23} + \delta_{32}| \equiv |a_{23}|$.

(ii) $dp_2/d\theta \geq 0$ *if the following two conditions hold:*

(b) $a_{21} \equiv \delta_{21} + \delta_{12} > \delta_{13} + \delta_{31} \equiv a_{31}$,
(c) $\partial g_2(\theta)/\partial\theta > \partial g_3(\theta)/\partial\theta$.

Proof. We have $B = \emptyset$ since every component of b is non-positive. Applying Propositions 2.3 and 2.4 to this problem yields the results. □

The inequality $dp_3/d\theta \geq 0$ has sufficient conditions which are similar to Conditions (b) and (c). Therefore, these sufficient conditions are omitted.

Condition (a) means that the complement effects between goods 2 and 3, in modulus, are relatively small, compared with each relevant substitution effect. Since, as mentioned above, these complement effects are the only effects that may prevent the optimal price of good 1 from being raised, it is intuitively plausible to have this condition.

Condition (b) means that an increase in the price of good 1 gives rise to a greater positive externality upon good 2 than upon good 3. It is thus plausible that this condition works in favor of good 2 since good 1 is assumed to have an increase in demand. It is easy to interpret Condition (c) since it means that the demand for good 2 is higher than that for good 3.

The following corollary studies the case where the demand for both goods 2 and 3 increases while the demand for good 1 decreases:

Corollary 3.4. *Consider the case where there exist three goods, and the demand for good 1 is decreased, i.e.* $\partial g_1(\theta)/\partial\theta < 0$, *but goods 2 and 3 have increases in demand, i.e.* $\partial g_i(\theta)/\partial\theta \geq 0$ *for* $i = 2, 3$. *Assume that* $|\delta_{jj}| > \sum_{i \neq j} |\delta_{ij}|$ *for every* $j = 1, 2, 3$, *and that* $|\delta_{ii}| > \sum_{j \neq i} |\delta_{ij}|$ *for every* $i = 1, 2, 3$. *We have* $dp_2/d\theta \geq 0$ *if the following two conditions hold:*

(i) $a_{12} \equiv \delta_{12} + \delta_{21} > \delta_{13} + \delta_{31} \equiv a_{13}$, *and*

(ii) $\partial g_2(\theta)/\partial\theta > \partial g_3(\theta)/\partial\theta + |\partial g_1(\theta)/\partial\theta|$.

Proof. In this case, $b_i < 0$ ($i = 2, 3$), and $b_1 > 0$. By applying Proposition 2.4 to this comparative statics problem, we can derive the above conditions. □

This corollary provides sufficient conditions guaranteeing that the optimal price of good 2 should be raised (because good 2 and good 3 are symmetric, good 3 has similar conditions). However, we cannot find sufficient conditions guaranteeing that the optimal price of good 1 should be raised. This inability is not surprising since, in this corollary, good 1 has an decrease in demand, compared with Corollary 3.3.

Condition (ii) has similar reasoning with Condition (c) of Corollary 3.3. Condition (i) is actually identical to Condition (b) of Corollary 3.3 since $a_{12} = a_{21}$ and $a_{13} = a_{31}$. However, Condition (i) is counter-intuitive since it is expected to work against $dp_2/d\theta \geq 0$ when good 1 has an decrease in demand, as is assumed in this corollary.

Appendix

1 Proof of Proposition 2.3

Here we take $i = n$. Assume that $x_n < 0$. It will be proved that this assumption leads to a contradiction.

It is assumed that $m (0 \leq m \leq n-1)$ entries of $x_1, x_2, \cdots, x_{n-1}$ are negative. For the case with $m = 0$, $x_h \geq 0$ for every $h \neq n$. The condition $a_{nj} \geq 0$ for every $j \neq n$, together with $a_{nn} < 0$, implies that the inner product of a_n and x is strictly positive, i.e., $(a_{n1}, \cdots, a_{nn}) \cdot (x_1, \cdots, x_n) > 0$. This contradicts $b_n \leq 0$, since b_n is equal to the inner product of a_n and x.

For the case where $m \neq 0$, it is assumed, without loss of generality, that $x_1, \cdots, x_m$ are negative, and that the other $n-1-m$ entries are nonnegative. The row operation, based on summing up a_n and the first m rows $a_1, \cdots, a_m$, yields

$$b_n + \sum_{h=1}^{m} b_h = \left(a_{n1} + \sum_{h=1}^{m} a_{h1}, \cdots, a_{nn} + \sum_{h=1}^{m} a_{hn} \right) \cdot (x_1, \cdots, x_n).$$

Note that the terms $a_{n1} + \sum_{h=1}^{m} a_{h1}, \cdots, a_{nm} + \sum_{h=1}^{m} a_{hm}$ and $a_{nn} + \sum_{h=1}^{m} a_{hn}$ include diagonals $a_{11}, \cdots, a_{mm}$ and a_{nn}, respectively. As a result, given that A is a negative dominant diagonal matrix, it follows that all of them are negative. Furthermore, Condition (i) implies that the other entries $a_{nj} + \sum_{h=1}^{m} a_{hj}$ $(j = m+1, \cdots, n-1)$ are nonnegative. Therefore, the inner product of the two vectors in the above equation is positive, and hence $b_n + \sum_{h=1}^{m} b_h > 0$. This contradicts $b_n + \sum_{h=1}^{m} b_h \leq 0$, which follows from Condition (ii). □

2 Proof of Proposition 2.4

We begin by presenting the following two lemmas before proceeding to prove Proposition 2.4.

Lemma A.1. *Assume that A has a dominant diagonal. If m $(0 \leq m \leq n)$ diagonals of A are negative, then the sign of the determinant of A is the same as the sign of $(-1)^m$.*

Proof. Let $J = \{j; a_{jj} < 0\}$ denote the index set of negative diagonals of A, and its number of members, $|J| = m$ $(0 \leq m \leq n)$. We define a corresponding positive dominant diagonal matrix C so that the j th column vector c^j is equal to $-a^j$ if $j \in J$ and to a^j otherwise. Then, $\det A = (-1)^m \det C$. Price (1951) demonstrated that for any positive dominant diagonal matrix C, $\det C > 0$. Clearly, the sign of $\det A$ is the same as the sign of $(-1)^m$. □

The original linear system $Ax = b$ can be examined by working with the augmented matrix:

$$[A|b] = \begin{pmatrix} a_{11} & \cdots & a_{1n} & | & b_1 \\ \vdots & \ddots & \vdots & | & \vdots \\ a_{n1} & \cdots & a_{nn} & | & b_n \end{pmatrix}. \tag{A.1}$$

Applying the Gaussian elimination for the first row to Equation (A.1) yields the row-reduced augmented matrix:[5]

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & | & b_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & | & b_n^{(1)} \end{pmatrix},$$

where every $i \geq 2$ and $j \geq 2$,

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} = \frac{\begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}}{a_{11}}, \text{ and } b_i^{(1)} = b_i - \frac{a_{i1}}{a_{11}} b_1 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{i1} & b_i \end{vmatrix}}{a_{11}}.$$

Similarly, applying the Gaussian elimination for the second row to the above augmented matrix yields the second-ordered augmented matrix

[5]The Gaussian elimination is the procedure for row-reducing the coefficient matrix to the reduced row echelon form. This operation can be easily checked in any textbook on linear algebra.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & | & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & | & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} & | & b_3^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} & | & b_n^{(2)} \end{pmatrix},$$

where for every $i \geq 3$ and $j \geq 3$,

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} a_{2j}^{(1)} = \frac{\begin{vmatrix} a_{22}^{(1)} & a_{2j}^{(1)} \\ a_{i2}^{(1)} & a_{ij}^{(1)} \end{vmatrix}}{a_{22}^{(1)}} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \text{ and}$$

$$b_i^{(2)} = b_i^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} b_2^{(1)} = \frac{\begin{vmatrix} a_{22}^{(1)} & b_2^{(1)} \\ a_{i2}^{(1)} & b_i^{(1)} \end{vmatrix}}{a_{22}^{(1)}} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{i1} & a_{i2} & b_i \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

By continuing this process step by step, an r-ordered augmented matrix,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1,r+1} & \cdots & a_{1n} & | & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2r}^{(1)} & a_{2,r+1}^{(1)} & \cdots & a_{2n}^{(1)} & | & b_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & \cdots & a_{rr}^{(r-1)} & a_{r,r+1}^{(r-1)} & \cdots & a_{rn}^{(r-1)} & | & b_r^{(r-1)} \\ 0 & 0 & \cdots & 0 & a_{r+1,r+1}^{(r)} & \cdots & a_{r+1,n}^{(r)} & | & b_{r+1}^{(r)} \\ \vdots & \vdots & \ddots & 0 & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & \cdots & 0 & a_{n,r+1}^{(r)} & \cdots & a_{nn}^{(r)} & | & b_n^{(r)} \end{pmatrix}, \quad \text{(A.2)}$$

is obtained.

In the following lemma, we will derive the values of the nonzero entries $a_{ij}^{(r)}$ and $b_i^{(r)}$ in Equation (A.2).

Lemma A.2. *According to Equation (A.2), the nonzero entries of the augmented matrix after the r-ordered Gaussian elimination are $a_{ij}^{(r)} = |A_{ij}^r|/|A^r|$ and $b_i^{(r)} = |A_{ib}^r|/|A^r|$ for every $1 \leq r \leq n-1$, $i \geq r+1$, and $j \geq r+1$. Here A^r stands* *for the first $r \times r$ submatrix of A:*

$$A^r = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}.$$

Furthermore, A^r_{ij} and A^r_{ib} denote the bordered matrices obtained from A^r by row $(a_{i1}, \cdots, a_{ir}, a_{ij})$ and column $(a_{1j}, \cdots, a_{rj}, a_{ij})$, and by row $(a_{i1}, \cdots, a_{ir}, b_i)$ and column $(b_1, \cdots b_r, b_i)$, respectively.

Proof. We prove that the nonzero entries in Equation (A.2) can be expressed as $a_{ij}^{(m)} = |A^m_{ij}|/|A^m|$, and $b_i^{(m)} = |A^m_{ib}|/|A^m|$. If $m = 1$, $a_{ij}^{(1)} = |A^1_{ij}|/|A^1|$, which is certainly true. Next, if we assume that $m-1$ is valid, then we would like to show that m is true. We obtain

$$a_{ij}^{(m)} = \frac{\begin{vmatrix} a_{mm}^{(m-1)} & a_{mj}^{(m-1)} \\ a_{im}^{(m-1)} & a_{ij}^{(m-1)} \end{vmatrix}}{a_{mm}^{(m-1)}} = \frac{\begin{vmatrix} |A^{m-1}_{mm}| & |A^{m-1}_{mj}| \\ |A^{m-1}_{im}| & |A^{m-1}_{ij}| \end{vmatrix}}{|A^{m-1}|\,|A^{m-1}_{mm}|}$$

$$= \frac{\begin{vmatrix} |A^{m-1}_{mm}| & -|A^{m-1}_{im}| \\ -|A^{m-1}_{mj}| & |A^{m-1}_{ij}| \end{vmatrix}}{|A^{m-1}|\,|A^{m-1}_{mm}|} = \frac{|A^m_{ij}|}{|A^m|},$$

where the last equality follows from Jacobi's ratio theorem (see Murata, 1977, p. 7). Similarly, we have $b_i^{(m)} = \begin{vmatrix} a_{mm}^{(m-1)} & b_m^{(m-1)} \\ a_{im}^{(m-1)} & b_i^{(m-1)} \end{vmatrix} \Big/ a_{mm}^{(m-1)} = |A^m_{ib}|/|A^m|$. □

Proof of Proposition 2.4

After establishing the above two lemmas, we are now in a position to prove Proposition 2.4. Here we take $i = n$. If we assume that $x_n < 0$, it will be proved that this assumption leads to a contradiction.

It is assumed, without loss of generality, that $R_n = \{1, 2, \cdots, r\}$ $(1 \leq r \leq n-1)$, i.e. $a_{nj} < 0 (j = 1, 2, \cdots, r)$. Only the signs of $x_{r+1}, \cdots, x_{n-1}$ are of concern because the first r entries of the row vector a_n will be replaced with zero elements by Gaussian elimination (recall that x_n has already been assumed to be negative). Assume that the first m entries of $x_{r+1}, \cdots, x_{n-1}$ are negative, and that the remaining $n-1-r-m$ entries are nonnegative $(m = 0, 1, \cdots, n-1-r)$. The row operation based on the sum of a_n and the first m rows $a_{r+1}, \cdots, a_{r+m}$ is

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1,r+1} & \cdots & a_{1n} & | & b_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{r,r+1} & \cdots & a_{rn} & | & b_r \\ a_{r+1,1} & \cdots & a_{r+1,r} & a_{r+1,r+1} & \cdots & a_{r+1,n} & | & b_{r+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{r+m,1} & \cdots & a_{r+m,r} & a_{r+m,r+1} & \cdots & a_{r+m,n} & | & b_{r+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,r} & a_{n-1,r+1} & \cdots & a_{n-1,n} & | & b_{n-1} \\ \alpha_{n1} & \cdots & \alpha_{nr} & \alpha_{n,r+1} & \cdots & \alpha_{nn} & | & \beta_n \end{pmatrix},$$

where $\alpha_{nj} = a_{nj} + \sum_{h=r+1}^{r+m} a_{hj}$, and $\beta_n = b_n + \sum_{h=r+1}^{r+m} b_h$. Applying the r th-ordered Gaussian elimination for this new augmented matrix, we can derive the values of the nonzero entries $\alpha_{nj}^{(r)} (j = r+1, \cdots, n)$ and $\beta_n^{(r)}$ by means of Lemma A.2:

$$\alpha_{nj}^{(r)} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1r} & a_{1j} \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rj} \\ \alpha_{n1} & \cdots & \alpha_{nr} & \alpha_{nj} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}} = \frac{\left|\hat{A}_{nj}^r\right|}{|A^r|} \text{ for every } j \geq r+1, \text{ and}$$

$$\beta_n^{(r)} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1r} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & b_r \\ \alpha_{n1} & \cdots & \alpha_{nr} & \beta_n \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}} = \frac{\left|\hat{A}_{nb}^r\right|}{|A^r|}.$$

The next step is to apply Lemma A.1 to study the signs of $\alpha_{nj}^{(r)}$ and $\beta_n^{(r)}$. A^r is a matrix with a dominant diagonal since A is assumed to be a dominant diagonal matrix. Furthermore, $\hat{A}_{nj}^r$ is also a dominant diagonal matrix if α_{nj} dominates $a_{1j}, a_{2j}, \cdots a_{rj}$, i.e. $|\alpha_{nj}| > \sum_{i=1}^{r} |a_{ij}|$. Therefore, Lemma A.1 implies that $\alpha_{nj}^{(r)}$ and α_{nj} share the same sign if α_{nj} dominates $a_{1j}, a_{2j}, \cdots a_{rj}$. Similarly, $\beta_n^{(r)}$ and β_n share the same sign if β_n dominates $b_1, \cdots, b_r$.

If $j = r+1, \cdots, r+m$, then $\alpha_{nj} = a_{jj} + a_{nj} + \sum_{\substack{h=r+1 \\ h \neq j}}^{r+m} a_{hj}$ includes a diagonal of A, namely, a_{jj}. This implies that α_{nj} is negative and that α_{nj} dominates $a_{1j}, a_{2j}, \cdots a_{rj}$. Therefore, we have $\alpha_{nj}^{(r)} < 0$. Similarly, $\alpha_{nn}^{(r)} < 0$, since α_{nn} includes a_{nn}.

If $j = r+m+1, \cdots, n-1$, then $j \notin R_n$. Therefore, Condition (i) implies that $\alpha_{nj} = a_{nj} + \sum_{h=r+1}^{r+m} a_{hj}$ is nonnegative and that α_{nj} dominates $a_{1j}, a_{2j}, \cdots a_{rj}$. As a result, $\alpha_{nj}^{(r)} \geq 0$. Condition (ii) implies that β_n is negative and that β_n dominates $b_1, \cdots, b_r$. Therefore, $\beta_n^{(r)} \leq 0$. However,

$$\beta_n^{(r)} = (0, \cdots, 0, \alpha_{n,r+1}^{(r)}, \cdots, \alpha_{n,r+m}^{(r)}, \alpha_{n,r+m+1}^{(r)}, \cdots, \alpha_{n,n-1}^{(r)}, \alpha_{n,n}^{(r)}) \cdot (x_1, \cdots, x_n) > 0.$$

This contradicts $\beta_n^{(r)} \leq 0$. □

Remark: We usually utilize the conventional Cramer's rule to solve a linear system like (1.1). Therefore, it is important to compare Cramer's rule with our method. The solution x_i obtained by Cramer's rule is equivalent to applying the Gaussian elimination for the other $n-1$ rows to A and then solving the equation $a_{ii}^{(n-1)} x_i = b_i^{(n-1)}$. One advantage of Proposition 2.4 is that it only requires applying the Gaussian elimination for r rows, where r is the number of R_i elements.

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正負非對角項並存下優勢對角矩陣非負解的探討

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線性體系在各領域的應用十分廣泛，譬如說經濟學中比較靜態的探討可以轉述成一個線性體系求解的問題。為保證線性體系的解為非負，文獻上通常假設該體系的係數矩陣是梅滋勒矩陣，也就是說矩陣中非對角項的係數設定為非負值。本文放鬆這個假設求得非負解的充分條件，並應用至多產品獨占廠商利潤極大化定價的探討。

關鍵詞: 優勢對角矩陣, 線性體系, 網路接續定價, 利潤最大化定價, 藍氏定價
JEL 分類代號: C62, L12, L32